Mathematics 100 Test #1B

Name: SOLUTIONS

Instructor: George Ballinger

Term: Fall, 2012

Section: ____________

Mark: $\frac{}{25}$

> The Sharp EL-531 calculator may be used on this test.
>
> You may not use L'Hôpital's Rule when evaluating limits.
>
> Show all of your work in the space provided.
>
> The number of marks for each question is indicated in brackets.

1. Evaluate $f(-4)$ given $f(x) = \begin{cases} 5+7x, & x \le -4 \\ 3-2x, & x > -4 \end{cases}$.

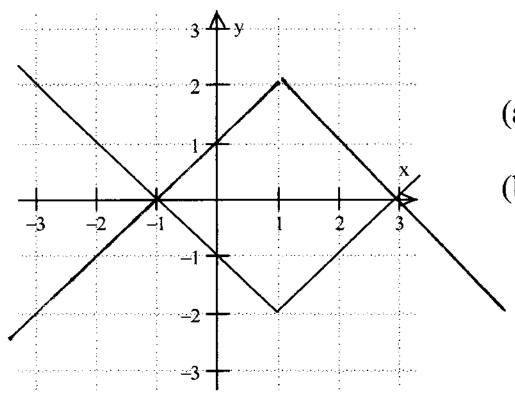
[1]

$$f(-4) = 5+7(-4) = 5-28 = -23$$

2. Given the graph of the function $y = f(x)$ below, (a) sketch the graph of $y = -f(x)$ in the same xy-plane and (b) state the domain of the derivative of $y = f(x)$.

reflect about x-axis

[2]

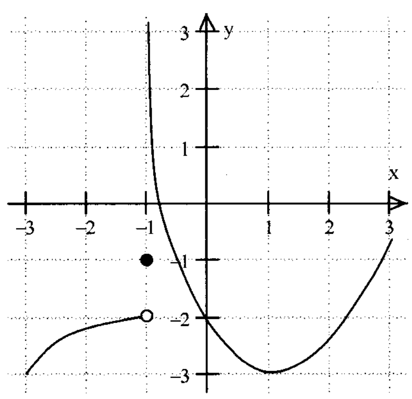


(a) see graph

(b) The domain of $f'(x)$ is $\{x \mid x \neq 1\}$.

3. Given the graph of the function $f(x)$ below, find $\lim\limits_{x \to -1^-} f(x)$, or if it does not exist, then determine whether it is ∞ or $-\infty$ or neither.

[1]



$$\boxed{\lim_{x \to -1^-} f(x) = -2}$$

4. Find the equation of the tangent line to the curve $y = 2x^3 - 5x + 6$ at $x = -3$. Express your answer in the slope-intercept form $y = mx + b$.

[3]

$$y' = 6x^2 - 5$$

At $x = -3$, $y = -33$ and $y' = 49$ (slope)

$$y + 33 = 49(x+3)$$

$$y + 33 = 49x + 147$$

$$y = 49x + 114$$

5. Evaluate the limits. If they do not exist, then determine whether they are ∞ or $-\infty$ or neither.

(a) $\lim_{x \to -2} \frac{x^2-4}{3x^2+5x-2}$ $= \lim_{x \to -2} \frac{(x+2)(x-2)}{(3x-1)(x+2)} = \lim_{x \to -2} \frac{x-2}{3x-1} = \frac{-4}{-7} = \frac{4}{7}$

[2]

(b) $\lim_{x \to 3}\left[\frac{3}{x-3} - \frac{x}{x-3}\right]$ $= \lim_{x \to 3} \frac{3-x}{x-3} = \lim_{x \to 3} -1 = -1$

[1]

(c) $\lim_{\theta \to 0} 2\theta \cot\theta$ $= \lim_{\theta \to 0} \frac{2\theta \cos\theta}{\sin\theta} = \lim_{\theta \to 0} \frac{\theta}{\sin\theta} \cdot \lim_{\theta \to 0} 2\cos\theta = 1 \cdot 2 = 2$

[1]

6. Use the **limit definition** of the derivative to find $f'(x)$, where $f(x) = -6 + \sqrt{4x-9}$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[-6+\sqrt{4(x+\Delta x)-9}\right] - \left[-6+\sqrt{4x-9}\right]}{\Delta x}$$

[4]

$$= \lim_{\Delta x \to 0} \frac{\sqrt{4(x+\Delta x)-9} - \sqrt{4x-9}}{\Delta x} \cdot \frac{\sqrt{4(x+\Delta x)-9} + \sqrt{4x-9}}{\sqrt{4(x+\Delta x)-9} + \sqrt{4x-9}}$$

$$= \lim_{\Delta x \to 0} \frac{4(x+\Delta x) - 9 - (4x-9)}{\Delta x\left(\sqrt{4(x+\Delta x)-9} + \sqrt{4x-9}\right)} = \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x\left(\sqrt{4(x+\Delta x)-9} + \sqrt{4x-9}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{4}{\sqrt{4(x+\Delta x)-9} + \sqrt{4x-9}} = \frac{4}{\sqrt{4x-9} + \sqrt{4x-9}} = \frac{4}{2\sqrt{4x-9}}$$

$$= \frac{2}{\sqrt{4x-9}}$$

7. Use derivative rules to calculate the derivatives of the following functions.

(a) $y = \sqrt[3]{t} - \dfrac{6}{\sqrt{t}} = t^{1/3} - 6t^{-1/2}$

$y' = \frac{1}{3}t^{-2/3} + 3t^{-3/2} = \dfrac{1}{3t^{2/3}} + \dfrac{3}{t^{3/2}}$

[2]

(b) $g(\theta) = 4\theta^3 \sec^2\theta$

$g'(\theta) = 4\theta^3 \cdot 2\sec\theta \cdot \sec\theta\tan\theta + 12\theta^2\sec^2\theta$

[2] $= 8\theta^3\sec^2\theta\tan\theta + 12\theta^2\sec^2\theta$

or $4\theta^2\sec^2\theta\,(2\theta\tan\theta + 3)$

(c) $h(x) = \dfrac{3x^4}{(x^3+7)^2}$ (Express your answer in factored form.)

$h'(x) = \dfrac{(x^3+7)^2(12x^3) - (3x^4)(2)(x^3+7)(3x^2)}{(x^3+7)^4}$

[3]

$= \dfrac{6x^3(x^3+7)\left[2(x^3+7) - 3x^3\right]}{(x^3+7)^4}$

$= \dfrac{6x^3(2x^3+14-3x^3)}{(x^3+7)^3} = \dfrac{6x^3(-x^3+14)}{(x^3+7)^3}$

8. State the product rule of differentiation and then prove it.

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)}$$

[3]

$$\frac{d}{dx}[f(x)g(x)] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\left(g(x+\Delta x) - g(x)\right) + g(x)\left(f(x+\Delta x) - f(x)\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x+\Delta x) \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} g(x) \cdot \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f(x)g'(x) + g(x)f'(x).$$