

The Sharp EL-531 calculator may be used on this test.  
You may not use L'Hôpital's Rule when evaluating limits.  
Show all of your work in the space provided.  
The number of marks for each question is indicated in brackets.

Mark:

25

1. Fill in the blanks with meaningful answers. No justification is required.

(a) The slope of a line perpendicular to the line  $y = -\frac{3}{5}x + \frac{17}{2}$  is  $\frac{5}{3}$ .

(b) If  $\lim_{x \rightarrow c} f(x) = f(c)$ , then the function  $f$  is said to be continuous at  $x = c$ .

[2]

(c) If  $f(-x) = -f(x)$  for all  $x$ , then the graph of  $f$  is symmetric about the origin.

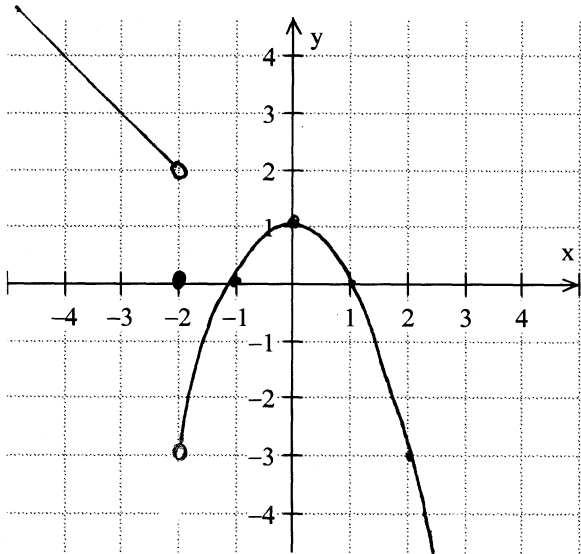
(d)  $\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = \underline{0}$ .

2. Consider the piecewise function

$$f(x) = \begin{cases} -x, & x < -2 \\ 0, & x = -2 \\ 1 - x^2, & x > -2 \end{cases}$$

Sketch the graph of  $f$  and evaluate the following limits.

[4]



(i)  $\lim_{x \rightarrow -2^+} f(x) = \underline{-3}$

(ii)  $\lim_{x \rightarrow -2^-} f(x) = \underline{2}$

(iii)  $\lim_{x \rightarrow -2} f(x) = \underline{\text{dne}}$

(iv)  $\lim_{x \rightarrow 0} f(x) = \underline{1}$

3. Evaluate each limit.

$$(a) \lim_{x \rightarrow 3} \left( \frac{1}{1 + \frac{1}{1+x}} \right) = \frac{1}{1 + \frac{1}{4}} = \frac{1}{5/4} = \frac{4}{5}$$

[1]

$$(b) \lim_{x \rightarrow 5} \frac{x^2 - x - 20}{2x^2 - 7x - 15} = \lim_{x \rightarrow 5} \frac{(x-5)(x+4)}{(2x+3)(x-5)} = \lim_{x \rightarrow 5} \frac{x+4}{2x+3} = \frac{9}{13}$$

[2]

4. Use the limit definition of the derivative to find  $f'(x)$ , where  $f(x) = -\sqrt{1+x}$ .

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-\sqrt{1+x+\Delta x} - (-\sqrt{1+x})}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-(\sqrt{1+x+\Delta x} - \sqrt{1+x})}{\Delta x} \cdot \frac{\sqrt{1+x+\Delta x} + \sqrt{1+x}}{\sqrt{1+x+\Delta x} + \sqrt{1+x}} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-[(1+x+\Delta x) - (1+x)]}{\Delta x (\sqrt{1+x+\Delta x} + \sqrt{1+x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-\Delta x}{\Delta x (\sqrt{1+x+\Delta x} + \sqrt{1+x})} \\ &= \lim_{\Delta x \rightarrow 0} \frac{-1}{\sqrt{1+x+\Delta x} + \sqrt{1+x}} \\ &= \frac{-1}{\sqrt{1+x} + \sqrt{1+x}} \\ &= \frac{-1}{2\sqrt{1+x}} \end{aligned}$$

[3]

5. Use derivative rules to calculate the derivatives of the following functions. Simplify your answers.

(a)  $f(x) = x^3 \sin x$

[1]  $f'(x) = x^3 \cos x + 3x^2 \sin x$

(b)  $y = 3 \csc^4(2\theta)$

[2] 
$$y' = 12 \csc^3(2\theta) [-\csc(2\theta) \cot(2\theta) \cdot 2]$$

$$= -24 \csc^4(2\theta) \cot(2\theta)$$

(c)  $h(x) = \frac{x^4}{(2x-1)^{3/2}}$

[3] 
$$h'(x) = \frac{(2x-1)^{3/2} (4x^3) - x^4 \left(\frac{3}{2}\right) (2x-1)^{1/2} (2)}{[(2x-1)^{3/2}]^2}$$

$$= \frac{4x^3 (2x-1)^{3/2} - 3x^4 (2x-1)^{1/2}}{(2x-1)^3}$$

$$= \frac{x^3 (2x-1)^{1/2} [4(2x-1) - 3x]}{(2x-1)^3}$$

$$= \frac{x^3 (8x - 4 - 3x)}{(2x-1)^{5/2}}$$

$$= \frac{x^3 (5x - 4)}{(2x-1)^{5/2}}$$

6. Find the coordinates of the point(s) (if any) at which the graph of  $f(x) = \frac{\sqrt{3}}{2}x + \sin x$ , for  $0 \leq x \leq \pi$ , has a horizontal tangent line.

$f'(x) = \frac{\sqrt{3}}{2} + \cos x = 0 \implies \cos x = -\frac{\sqrt{3}}{2} \implies x = \frac{5\pi}{6}$   
 for  $0 \leq x \leq \pi$   
 Need zero slope (derivative)

[3]  $f\left(\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{2} \cdot \frac{5\pi}{6} + \sin \frac{5\pi}{6} = \frac{5\pi\sqrt{3}}{12} + \frac{1}{2}$

$\therefore$  point is  $\left(\frac{5\pi}{6}, \frac{5\pi\sqrt{3}}{12} + \frac{1}{2}\right)$

7. State the quotient rule of differentiation and then prove it using the limit definition of a derivative.

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \lim_{\Delta x \rightarrow 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x) - f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}$$

[4]

$$= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{g(x) \frac{f(x+\Delta x) - f(x)}{\Delta x} - f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x}}{g(x+\Delta x)g(x)}$$

$$= \frac{g(x) \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} - f(x) \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}}{\lim_{\Delta x \rightarrow 0} [g(x+\Delta x)g(x)]}$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{[g(x)]^2}$$