Mathematics 100 Test #1A Name: SOLUTIONS

Instructor: George Ballinger **Section:** ________

Term: Fall, 2018

Mark: ___ / 25

The Sharp EL-531 calculator may be used on this test.

You may not use L'Hôpital's Rule when evaluating limits.

Show all of your work in the space provided.

The number of marks for each question is indicated in brackets.

1. Fill in the blanks with meaningful answers. No justification is required.

[2]

(a) The domain of $f(x)=\dfrac{7}{\sqrt{x-9}}$ is $(9, \infty)$.

(b) According to the Intermediate Value Theorem, if a function f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there exists some c in $[a, b]$ such that $f(c) = K$.

(c) The graph of an even function is symmetric about the y-axis.

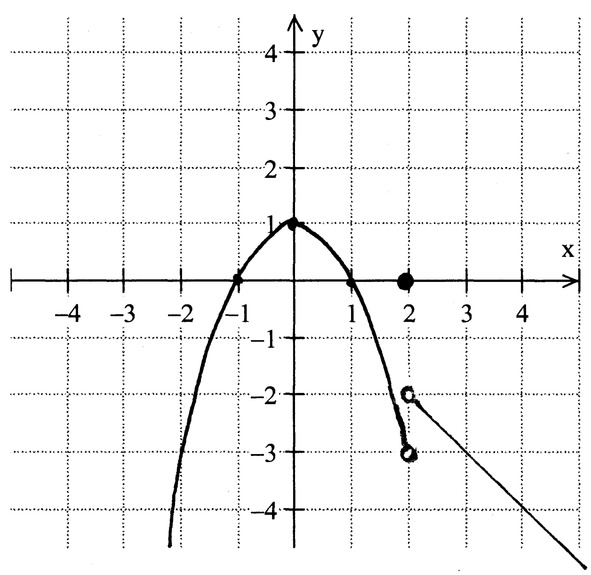
(d) $\lim\limits_{x \to 0} \dfrac{x}{\sin x} =$ 1.

2. Consider the piecewise function

$$f(x)=\begin{cases} 1-x^2, & x<2 \\ 0, & x=2 \\ -x, & x>2 \end{cases}.$$

Sketch the graph of f and evaluate the following limits.

[4]



(i) $\lim\limits_{x \to 2^+} f(x) =$ -2

(ii) $\lim\limits_{x \to 2^-} f(x) =$ -3

(iii) $\lim\limits_{x \to 2} f(x) =$ dne

(iv) $\lim\limits_{x \to 0} f(x) =$ 1

3. Evaluate each limit.

(a) $\lim_{x\to 1}\left(\dfrac{1}{1+\dfrac{1}{1+x}}\right) = \dfrac{1}{1+\frac{1}{2}} = \dfrac{1}{3/2} = \dfrac{2}{3}$

[1]

(b) $\lim_{x\to 5}\dfrac{x^2-6x+5}{2x^2-3x-35} = \lim_{x\to 5}\dfrac{(x-5)(x-1)}{(2x+7)(x-5)} = \lim_{x\to 5}\dfrac{x-1}{2x+7} = \dfrac{4}{17}$

[2]

4. Use the limit definition of the derivative to find $f'(x)$, where $f(x)=\sqrt{1-x}$.

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sqrt{1-(x+\Delta x)}-\sqrt{1-x}}{\Delta x}\cdot\frac{\sqrt{1-(x+\Delta x)}+\sqrt{1-x}}{\sqrt{1-(x+\Delta x)}+\sqrt{1-x}}$$

[3]

$$= \lim_{\Delta x\to 0}\frac{[1-(x+\Delta x)]-(1-x)}{\left[\sqrt{1-(x+\Delta x)}+\sqrt{1-x}\right]\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-\Delta x}{\left[\sqrt{1-(x+\Delta x)}+\sqrt{1-x}\right]\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-1}{\sqrt{1-(x+\Delta x)}+\sqrt{1-x}}$$

$$= \frac{-1}{\sqrt{1-x}+\sqrt{1-x}}$$

$$= \frac{-1}{2\sqrt{1-x}}$$

5. Use derivative rules to calculate the derivatives of the following functions. Simplify your answers.

(a) $f(x) = x^7 \cos x$

[1] $f'(x) = -x^7 \sin x + 7x^6 \cos x$

(b) $y = -8\sec^2(3x)$

$y' = -16\sec(3x) \cdot \sec(3x)\tan(3x) \cdot 3$

[2] $= -48\sec^2(3x)\tan(3x)$

(c) $h(t) = \dfrac{t^2}{(6t-5)^{3/2}}$

$$h'(t) = \frac{(6t-5)^{3/2}(2t) - t^2\left(\frac{3}{2}\right)(6t-5)^{1/2}(6)}{\left[(6t-5)^{3/2}\right]^2}$$

$$= \frac{2t(6t-5)^{3/2} - 9t^2(6t-5)^{1/2}}{(6t-5)^3}$$

[3]
$$= \frac{t(6t-5)^{1/2}\left[2(6t-5) - 9t\right]}{(6t-5)^3}$$

$$= \frac{t(12t-10-9t)}{(6t-5)^{5/2}}$$

$$= \frac{t(3t-10)}{(6t-5)^{5/2}}$$

6. Find the coordinates of the point(s) (if any) at which the graph of $f(x) = x + 2\sin x$, for $0 \le x \le \pi$, has a horizontal tangent line.

Need zero slope (derivative)

$f'(x) = 1 + 2\cos x = 0 \Rightarrow \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}$ for $0 \le x \le \pi$

$f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + \sqrt{3}$

[3]

$\therefore$ point is $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right)$

7. State the quotient rule of differentiation and then prove it using the limit definition of a derivative.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \boxed{\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x\to 0}\frac{\frac{f(x+\Delta x)}{g(x+\Delta x)}-\frac{f(x)}{g(x)}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}$$

[4]

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{g(x)\frac{f(x+\Delta x)-f(x)}{\Delta x}-f(x)\frac{g(x+\Delta x)-g(x)}{\Delta x}}{g(x+\Delta x)g(x)}$$

$$= \frac{g(x)\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}-f(x)\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}}{\lim_{\Delta x\to 0}[g(x+\Delta x)g(x)]}$$

$$= \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$