Riemann Sum and Definite Integral

Definition of Riemann Sum

Let f be defined on $[a, b]$ and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b,$$

where $\Delta x_i = x_i - x_{i-1}$ is the width of the i^{th} subinterval $[x_{i-1}, x_i]$. If c_i is *any* point in $[x_{i-1}, x_i]$, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i,$$

is called a **Riemann sum** of f for the partition Δ.

Norms of Partitions

The width of the largest subinterval $[x_{i-1}, x_i]$ of a partition Δ is called the **norm** of Δ and is denoted $\|\Delta\|$. In other words,

$$\|\Delta\| = \max\{\Delta x_1, \Delta x_2, \Delta x_3, \ldots, \Delta x_n\} = \max\{x_1 - x_0, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}\}.$$

If a partition is **regular**, meaning all the subintervals have the same width Δx, then

$$\|\Delta\| = \Delta x = \frac{b-a}{n}.$$

Definition of Definite Integral

Let f be defined on $[a, b]$. If the limit of Riemann sums over partitions Δ exists as the norm of Δ approaches zero, then f is said to be **integrable** on $[a, b]$, and the limit is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i,$$

which is called the **definite integral** of f from a to b. The numbers a and b are called **lower** and **upper limits of integration**, respectively. In the case of a regular partition and if right endpoints are used (i.e. $c_i = x_i = a + i\Delta x$), this becomes

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i\Delta x)\Delta x.$$