# Numerical Integration

Let $f$ be a continuous function on $[a,b]$. Finding the exact value of

$$\int_a^b f(x)\,dx,$$

generally requires first finding an antiderivative $F$ for $f$ and then applying the Fundamental Theorem of Calculus and computing $F(b) - F(a)$. Unfortunately, the antiderivatives of many functions cannot be expressed in terms of elementary functions (in other words they cannot be "found"). Even when antiderivatives are elementary, they may be hard to come by. We therefore resort to numerical means of approximating definite integrals.

To approximate

$$\int_a^b f(x)\,dx,$$

one can divide $[a,b]$ into $n$ subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/n$ and use rectangles to approximate the integral. Using left endpoints $x_{i-1} = a + (i-1)\Delta x$ gives

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Using right endpoints $x_i = a + i\Delta x$ gives

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\Delta x.$$

Using the midpoints $(x_{i-1}+x_i)/2 = a + (i-1/2)\Delta x$ of the subintervals $[x_{i-1}, x_i]$ gives

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x,$$

which is known as the **Midpoint Rule**.

When using rectangles, constant functions (i.e. zero-degree polynomials) are used to approximate $f$ on each subinterval $[x_{i-1}, x_i]$. A better approach is to use first-degree polynomials (secant lines), which creates trapezoidal regions, or second-degree polynomials (parabolic curves). These give rise to the **Trapezoidal Rule** and **Simpson's Rule**, respectively, for approximating definite integrals.

**Trapezoidal Rule**

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

**Simpson's Rule** (assuming $n$ is even)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)].$$

In each case, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\,dx$.