Interest

Suppose an initial principal amount P is invested at an annual interest rate r. Then after t years the amount (or balance) A of the investment will be

1. $A = P(1+rt)$, if interest is calculated only on the principal amount (i.e. simple interest)
2. $A = P\left(1+\frac{r}{n}\right)^{nt}$, if interest is compounded n times per year
   (a) $A = P(1+r)^t$, if interest is compounded annually ($n=1$)
   (b) $A = P\left(1+\frac{r}{4}\right)^{4t}$, if interest is compounded quarterly ($n=4$)
   (c) $A = P\left(1+\frac{r}{12}\right)^{12t}$, if interest is compounded monthly ($n=12$)
   (d) $A = P\left(1+\frac{r}{365}\right)^{365t}$, if interest is compounded daily ($n=365$, ignoring leap years)
3. $A = Pe^{rt}$, if interest is compounded continuously

Example: Suppose $1,000 is invested at a rate of 6% for three years, i.e. $P=1000$, $r=0.06$ and $t=3$. Then after three years the balance A will be

$A = 1000(1+(0.06)(3)) = \$1{,}180.00$,	if interest is simple
$A = 1000(1+0.06)^3 = \$1{,}191.02$,	if interest is compounded annually (6% per year)
$A = 1000(1+0.015)^{12} = \$1{,}195.62$,	if interest is compounded quarterly (1.5% per quarter)
$A = 1000(1+0.005)^{36} = \$1{,}196.68$,	if interest is compounded monthly (0.5% per month)
$A = 1000(1+0.000164384)^{1095} = \$1{,}197.20$,	if interest is compounded daily (0.0164384% per day)
$A = 1000e^{(0.06)(3)} = \$1{,}197.22$,	if interest is compounded continuously

In the case of continuously compounded interest, the rate at which the investment grows is directly proportional to the value of the investment. Growth is therefore governed by the differential equation

$$\frac{dA}{dt} = rA, \text{ with initial condition } A(0) = P,$$

where the constant of proportionality is the interest rate r.

Note also that $P\left(1+\frac{r}{n}\right)^{nt} \to Pe^{rt}$ as $n \to \infty$ since

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = P\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{(n/r)rt} = P\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{xrt} = P\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{rt} = Pe^{rt},$$

where we replaced n/r by x and observed that $x \to \infty$ as $n \to \infty$.