Differentials and Error Propagation

Suppose some quantity, x, is measured. There is usually some unknown error, Δx, associated with the measurement, where the exact value is $x + \Delta x$. Computations based on x, e.g. $y = f(x)$, will therefore have some error

$$\Delta y = f(x + \Delta x) - f(x),$$

called the **propagated error**. Here Δy is a measure of the **absolute error**, whereas $\Delta y/y$ represents the **relative error** and $\Delta y/y \cdot 100\%$ is the **percent error**.

For example, suppose the radius of a circle is measured to be $r = 6$ cm but the exact value of the radius (if it could be measured with infinite precision) is 6.15 cm. Then $\Delta r = 0.15$ cm. Computing the area of the circle using the formula $A = \pi r^2$ based on the measurement of $r = 6$ cm gives $A = \pi 6^2 \approx 113.097$ cm^2, whereas the exact area should be $A = \pi 6.15^2 \approx 118.823$ cm^2. The difference,

$$\Delta A = A(6.15) - A(6) \approx 5.726 \text{ cm}^2,$$

is the error in the computed area due to the fact that an approximation of the exact radius was used.

Differentials can be used to approximate propagated errors. If $y = f(x)$, then

$$\Delta y \approx dy = f'(x)dx,$$

where $dx = \Delta x$.

In the previous example

$$\Delta A \approx dA = 2\pi r dr = 2\pi(6 \text{ cm})(0.15 \text{ cm}) \approx 5.655 \text{ cm}^2.$$

Therefore, the propagated error is approximately 5.655 cm^2. Meanwhile the relative error, which gives a more meaningful measure of the relative size (how large or how small) the error is, is

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2dr}{r} = \frac{2(0.15)}{6} = 0.05,$$

or approximately 5%.

"Cow-culus" Exercise: A cow's udder is in the shape of a hemisphere.

1. If its diameter is measured to be 26 cm with a possible error of 0.5 cm, then use differentials to approximate the

   (a) propagated error

   (b) relative error

   (c) percent error

   in computing its volume.

2. Estimate the maximum allowable percent error in measuring the diameter if the error in computing the volume cannot exceed 3%.

Note: In this exercise the diameter is the quantity being measured, while the volume is being computed. We should therefore express the volume as a function of the diameter.