Continuity and the Intermediate Value Theorem

Definition of Continuity:

1. **Continuity at a Point**
   A function f is **continuous at** $\boldsymbol{c}$ if $\lim_{x \to c} f(x) = f(c)$.

   Note: This equality implies three things, that the limit of f exists at c, that f is defined at c, and that the limit and function values at c are equal.

2. **Continuity on an Open Interval**
   A function f is **continuous on an open interval** $\boldsymbol{(a, b)}$ if f is continuous at each point in the interval.

   Note: This definition also includes open intervals of the form $(-\infty, b)$, (a, ∞) and $(-\infty, \infty)$.

3. **Right Continuity at a Point**
   A function f is **right continuous at** $\boldsymbol{c}$ if $\lim_{x \to c^+} f(x) = f(c)$.

4. **Left Continuity at a Point**
   A function f is **left continuous at** $\boldsymbol{c}$ if $\lim_{x \to c^-} f(x) = f(c)$.

5. **Continuity on a Closed Interval**
   A function f is **continuous on a closed interval** $\boldsymbol{[a, b]}$ if f is continuous on the open interval (a, b), f is right continuous at a, and f is left continuous at b.

   Note: Similar definitions can be made to describe continuity on intervals of the form $(a, b]$, $[a, b)$, $(-\infty, b]$, and $[a, \infty)$.

Properties of Continuity: If f and g are continuous at c, then the following are also continuous at c: (i) bf, for any real number b, (ii) $f + g$, (iii) $f - g$, (iv) fg, and (v) f/g, if $g(c) \neq 0$.

Continuity of a Composite Function: If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Intermediate Value Theorem: If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there exists at least one number c in $[a, b]$ such that $f(c) = k$.