Analyzing and Graphing Functions

Let f be a function.

1. Find the domain of f. The domain must exclude x values that would cause division by zero or that would involve taking square roots (or other even roots) of negative numbers.

2. Find the y-intercept by evaluating $f(0)$ and plot the point.

3. Find any x-intercepts by solving $f(x) = 0$ and plot the point(s).

4. Check for symmetry of the graph about the y-axis (if $f(-x) = f(x)$ for all x) or the origin (if $f(-x) = -f(x)$ for all x).

5. Check whether the function is periodic (e.g. if it involves trigonometric functions) and identify its period.

6. Identify points of discontinuity of f and distinguish between removable versus nonremovable discontinuities. Find any vertical asymptotes and draw them on your graph by using dotted lines. Identify the left- and right-hand limiting behaviour of f at these asymptotes.

7. Examine the end behaviour of f by evaluating $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Find all horizontal and slant asymptotes and draw them on your graph by using dotted lines. Note that rational functions have horizontal asymptotes if the degree of the numerator is less than or equal to the degree of the denominator and they have slant asymptotes (found by long division) if the degree of the numerator is one more than the degree of the denominator.

Precalculus ↑

Calculus ↓

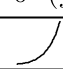
8. Calculate $f'(x)$ and find critical numbers, i.e. x values in the domain of f where $f'(x) = 0$ or $f'(x)$ is undefined.

9. Find intervals on which f is increasing or decreasing by examining the sign of $f'(x)$ between critical numbers and discontinuities of f and find relative extrema by applying the First Derivative Test. Plot these points.

10. Calculate $f''(x)$ and find where $f''(x) = 0$ or $f''(x)$ is undefined.

11. Find intervals on which the graph of f is concave upward or concave downward by examining the sign of $f''(x)$ between x values at which $f''(x) = 0$ or $f''(x)$ is undefined. Find and plot inflection points, i.e. points where the concavity changes and at which the graph has a tangent line. Apply the Second Derivative Test (optional) to verify relative extrema.

12. Plot extra points as needed. Connect points with a smooth curve (except where f is undefined). Depending on the sign of $f'(x)$ and $f''(x)$ on a given interval, the graph of f should have one of the following shapes:

	$f'(x) > 0 \ \ (f \nearrow)$	$f'(x) < 0 \ \ (f \searrow)$
$f''(x) > 0 \ \ (f \smile)$		
$f''(x) < 0 \ \ (f \frown)$		

Example 1: Analyze and graph $f(x) = \dfrac{x^2-4}{2x^2-2}$.

1. Re-writing $f(x)$ by factoring gives

$$f(x) = \frac{(x+2)(x-2)}{2(x+1)(x-1)}.$$

   The domain of f is therefore $\{x|\, x \neq \pm 1\}$.

2. Since $f(0) = 2$, then the y-intercept is $(0, 2)$.

3. If $f(x) = 0$, then $x = \pm 2$ and so $(2, 0)$ and $(-2, 0)$ are the x-intercepts.

4. Since

$$f(-x) = \frac{(-x)^2-4}{2(-x)^2-2} = \frac{x^2-4}{2x^2-2} = f(x),$$

   for all x, then f is even and its graph is symmetric about the y-axis.

5. f is not a periodic function.

6. f is continuous everywhere except at $x = \pm 1$, which are nonremovable discontinuities. The lines $x = 1$ and $x = -1$ are vertical asymptotes. Moreover,

$$\lim_{x\to -1^-} f(x) = \frac{-3}{0^+} = -\infty, \qquad \lim_{x\to -1^+} f(x) = \frac{-3}{0^-} = \infty,$$
$$\lim_{x\to 1^-} f(x) = \frac{-3}{0^-} = \infty, \qquad \lim_{x\to 1^+} f(x) = \frac{-3}{0^+} = -\infty.$$

7. Since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1-\dfrac{4}{x^2}}{2-\dfrac{2}{x^2}} = \frac{1}{2},$$

   then $y = 1/2$ is a horizontal asymptote.

Precalculus ↑

Calculus ↓

8. Calculating $f'(x)$ by using the quotient rule gives us

$$f'(x) = \frac{(2x^2-2)(2x)-(x^2-4)(4x)}{(2x^2-2)^2} = \frac{4x^3-4x-4x^3+16x}{4(x^2-1)^2}$$
$$= \frac{12x}{4(x^2-1)^2} = \frac{3x}{(x^2-1)^2} = \frac{3x}{(x+1)^2(x-1)^2},$$

   which is defined everywhere that f itself is, in other words $x \neq \pm 1$. If $f'(x) = 0$, then $x = 0$, which is the only critical number of f.

9.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f'(x)$	$-$	$-$	$+$	$+$
f Increasing or Decreasing	↘	↘	↗	↗

   The function f is decreasing on $(-\infty, -1)$ and $(-1, 0)$ and is increasing on $(0, 1)$ and $(1, \infty)$. By the First Derivative Test, f has a relative minimum at $(0, 2)$ but no relative maxima.

10. Calculating $f''(x)$ gives us

$$f''(x) = \frac{(x^2-1)^2(3)-(3x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{3(x^2-1)^2-12x^2(x^2-1)}{(x^2-1)^4}$$
$$= \frac{3(x^2-1)((x^2-1)-4x^2)}{(x^2-1)^4} = \frac{3(-3x^2-1)}{(x^2-1)^3} = \frac{-3(3x^2+1)}{(x^2-1)^3} = \frac{-3(3x^2+1)}{(x+1)^3(x-1)^3},$$

which is defined everywhere that f itself is, in other words $x \neq \pm 1$. Moreover, $f''(x)$ can never equal zero.

11.

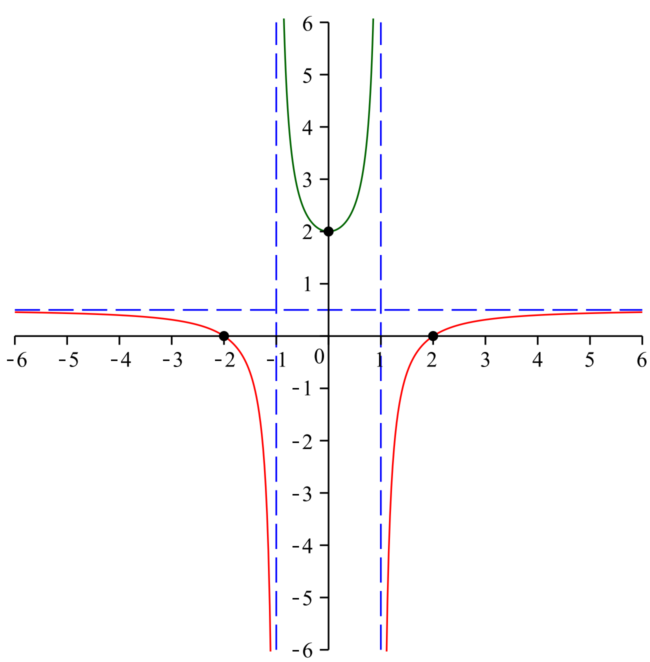
Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Sign of $f''(x)$	$-$	$+$	$-$
Concavity of f	$\frown$	$\smile$	$\frown$

The graph of f is concave upward on the interval $(-1,1)$ and it is concave downward on the intervals $(-\infty,-1)$ and $(1,\infty)$. The graph of f has no inflection points since the concavity changes only at $x=\pm 1$, which are points of discontinuity of the function f.

Since $f''(0) = 3 > 0$, then the Second Derivative Test confirms that f has a relative minimum at $(0,2)$.

12. Summary and Graph

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f'(x)$	$-$	$-$	$+$	$+$
f Increasing or Decreasing	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$
Sign of $f''(x)$	$-$	$+$	$+$	$-$
Concavity of f	$\frown$	$\smile$	$\smile$	$\frown$
Shape of f				



Example 2: Analyze and graph $f(x) = 2\cos x + \sin 2x$.

1. The domain of f is clearly $(-\infty, \infty)$.
2. Since $f(0) = 2$, then the y-intercept is $(0, 2)$.
3. Re-writing $f(x)$ gives us
$$f(x) = 2\cos x + 2\sin x \cos x = 2\cos x(1 + \sin x).$$
Therefore, if $f(x) = 0$, then $\cos x = 0$ or $\sin x = -1$. On the interval $[0, 2\pi]$ this means $x = \pi/2$ or $x = 3\pi/2$ and so f has x-intercepts $(\pi/2, 0)$ and $(3\pi/2, 0)$.
4. Since $f(-x) = 2\cos(-x) + \sin(2(-x)) = 2\cos x - \sin 2x$, which is neither $f(x)$ nor $-f(x)$, then f does not have symmetry about the y-axis or the origin.
5. Since $f(x + 2\pi) = f(x)$ for all x, then f is periodic with a period of 2π.
6. f is continuous everywhere and has no vertical asymptotes.
7. Since f is periodic, then it will have no horizontal or slant asymptotes and the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ will not exist.

Precalculus ↑

Calculus ↓

8. Calculating $f'(x)$ gives us
$$\begin{aligned} f'(x) &= -2\sin x + 2\cos 2x = -2\sin x + 2(1 - 2\sin^2 x) = -2(2\sin^2 x + \sin x - 1) \\ &= -2(2\sin x - 1)(\sin x + 1), \end{aligned}$$
which is defined everywhere. If $f'(x) = 0$, then $\sin x = 1/2$ or $\sin x = -1$. Therefore on the interval $[0, 2\pi]$, $x = \pi/6$, $5\pi/6$ and $3\pi/2$ are critical numbers.
9.

Interval	$(0, \pi/6)$	$(\pi/6, 5\pi/6)$	$(5\pi/6, 3\pi/2)$	$(3\pi/2, 2\pi)$
Sign of $f'(x)$	+	−	+	+
f Increasing or Decreasing	↗	↘	↗	↗

The function f is increasing on $(0, \pi/6)$ and on $(5\pi/6, 2\pi)$ and decreasing on $(\pi/6, 5\pi/6)$. By the First Derivative Test, f has a relative maximum at $(\pi/6, 3\sqrt{3}/2)$ and a relative minimum at $(5\pi/6, -3\sqrt{3}/2)$. At $(3\pi/2, 0)$, f has a horizontal tangent but neither a relative maximum nor a relative minimum.
10. Calculating $f''(x)$ gives us
$$f''(x) = -2\cos x - 4\sin 2x = -2\cos x - 8\sin x\cos x = -2\cos x(1 + 4\sin x),$$
which is defined everywhere. Solving $f''(x) = 0$ gives us $\cos x = 0$ or $\sin x = -1/4$ from which we get $x = \pi/2$, $x = 3\pi/2$, $x = \pi + \arcsin(1/4) \approx 3.39$ or $x = 2\pi - \arcsin(1/4) \approx 6.03$.
11.

Interval	$(0, \pi/2)$	$(\pi/2, 3.39)$	$(3.39, 3\pi/2)$	$(3\pi/2, 6.03)$	$(6.03, 2\pi)$
Sign of $f''(x)$	−	+	−	+	−
Concavity of f	⌢	⌣	⌢	⌣	⌢

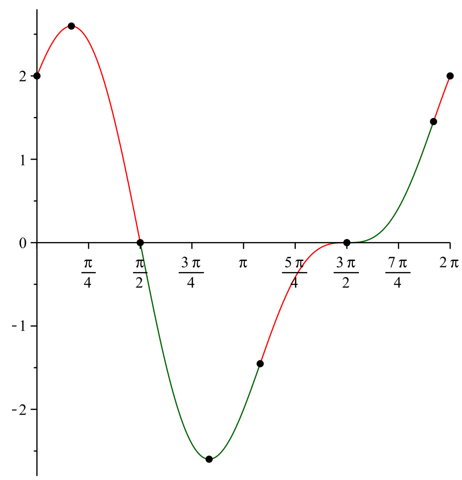
The graph of f is concave upward on the intervals $(\pi/2, 3.39)$ and $(3\pi/2, 6.03)$ and it is concave downward on the intervals $(0, \pi/2)$, $(3.39, 3\pi/2)$ and $(6.03, 2\pi)$. The graph of f has inflection points at $(\pi/2, 0)$, $(3\pi/2, 0)$, $(3.39, -1.45)$ and $(6.03, 1.45)$.

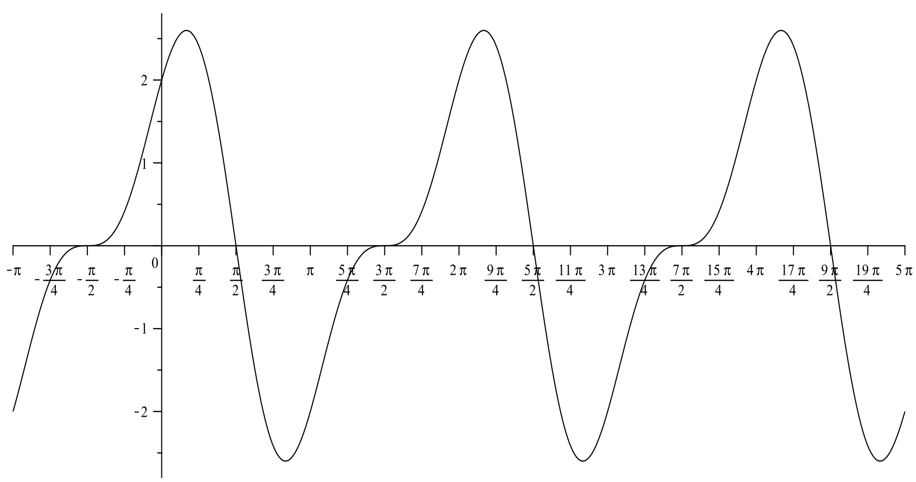
Since $f''(\pi/6) = -3\sqrt{3} < 0$ and $f''(5\pi/6) = 3\sqrt{3} > 0$, then the Second Derivative Test confirms our conclusions about the relative maximum and relative minimum. Note that $f''(3\pi/2) = 0$, so the Second Derivative Test would be unable to explain the behaviour of f at $x = 3\pi/2$.

Remark: At the endpoints of $[0, 2\pi]$, $f(0) = f(2\pi) = 2$. Therefore on the interval $[0, 2\pi]$, $\pm 3\sqrt{3}/2 \approx \pm 2.60$ are the absolute maximum and minimum values of f. Since f is periodic, then $\pm 3\sqrt{3}/2$ are the absolute maximum and minimum values of f on its domain of $(-\infty, \infty)$. Therefore the range of f is $[-3\sqrt{3}/2, 3\sqrt{3}/2]$.

12. Summary and Graph

Interval	$(0, \pi/6)$	$(\pi/6, \pi/2)$	$(\pi/2, 5\pi/6)$	$(5\pi/6, 3.39)$	$(3.39, 3\pi/2)$	$(3\pi/2, 6.03)$	$(6.03, 2\pi)$
Sign of $f'(x)$	+	−	−	+	+	+	+
f Incr. or Decr.	↗	↘	↘	↗	↗	↗	↗
Sign of $f''(x)$	−	−	+	+	−	+	−
Concavity of f	⌒	⌒	⌣	⌣	⌒	⌣	⌒
Shape of f	╭	╮	╰	╯	╭	╯	╭





Example 3: Analyze and graph $f(x) = x^{5/3} - 5x^{2/3}$.

1. The domain of f is clearly $(-\infty, \infty)$.
2. Since $f(0) = 0$, then the y-intercept is $(0,0)$.
3. Factoring $f(x)$ gives us
$$f(x) = x^{2/3}(x-5).$$
Therefore, if $f(x) = 0$, then $x = 0$ or $x = 5$ and so the x-intercepts are $(0,0)$ and $(5,0)$.
4. Since $f(-x) = (-x)^{5/3} - 5(-x)^{2/3} = -x^{5/3} - 5x^{2/3}$, which is neither $f(x)$ nor $-f(x)$, then f does not have symmetry about the y-axis or the origin.
5. f is not a periodic function.
6. f is continuous everywhere and has no vertical asymptotes.
7. The end behaviour of $f(x)$ is given by
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^{2/3}(x-5) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^{2/3}(x-5) = \infty.$$
There are neither horizontal nor slant asymptotes.

Precalculus ↑

Calculus ↓

8. Calculating $f'(x)$ gives us
$$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) = \frac{5(x-2)}{3x^{1/3}},$$
which is defined for all $x \neq 0$. If $f'(x) = 0$, then $x = 2$. Therefore both $x = 0$ (where $f'(x)$ is undefined) and $x = 2$ (where $f'(x) = 0$) are critical numbers of f.
9.

Interval	$(-\infty, 0)$	$(0,2)$	$(2,\infty)$
Sign of $f'(x)$	$+$	$-$	$+$
f Increasing or Decreasing	↗	↘	↗

The function f is increasing on $(-\infty, 0)$ and $(2,\infty)$ and is decreasing on $(0,2)$. By the First Derivative Test, f has a relative maximum at $(0,0)$ and a relative minimum at $(2, -3\sqrt[3]{4}) \approx (2, -4.76)$.
10. Calculating $f''(x)$ gives us
$$f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) = \frac{10(x+1)}{9x^{4/3}},$$
which is defined everywhere except at $x = 0$ where the function f has no tangent line (since $f'(0)$ is undefined). Clearly $f''(x) = 0$ at $x = -1$ and $f''(x)$ is undefined at $x = 0$.
11.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, \infty)$
Sign of $f''(x)$	$-$	$+$	$+$
Concavity of f	⌢	⌣	⌣

The graph of f is concave downward on the interval $(-\infty, -1)$ and it is concave upward on the intervals $(-1, 0)$ and $(0, \infty)$. The graph of f has an inflection point at the point $(-1, -6)$, where the concavity changes from downwards to upwards.

Since $f''(2) \approx 1.32 > 0$, then the Second Derivative Test confirms that f has a relative minimum at $x = 2$. Clearly the Second Derivative Test cannot be used to check whether f has a relative maximum or minimum at $x = 0$.

12. Summary and Graph

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f'(x)$	$+$	$+$	$-$	$+$
f Increasing or Decreasing	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$
Sign of $f''(x)$	$-$	$+$	$+$	$+$
Concavity of f	$\frown$	$\smile$	$\smile$	$\smile$
Shape of f